TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 355, Number 5, Pages 1857–1867
S 0002-9947(02)03243-9
Article electronically published on December 9, 2002

HYPERBOLIC 2-SPHERES WITH CONICAL SINGULARITIES, ACCESSORY PARAMETERS AND KÄHLER METRICS ON $\mathcal{M}_{0,n}$

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ABSTRACT. We show that the real-valued function S_α on the moduli space $\mathcal{M}_{0,n}$ of pointed rational curves, defined as the critical value of the Liouville action functional on a hyperbolic 2-sphere with $n \geq 3$ conical singularities of arbitrary orders $\alpha = \{\alpha_1, \dots, \alpha_n\}$, generates accessory parameters of the associated Fuchsian differential equation as their common antiderivative. We introduce a family of Kähler metrics on $\mathcal{M}_{0,n}$ parameterized by the set of orders α, explicitly relate accessory parameters to these metrics, and prove that the functions S_α are their Kähler potentials.

1. Introduction

The existence and uniqueness of a hyperbolic metric (a conformal metric of constant negative curvature -1) with prescribed singularities at a finite number of points on a Riemann surface is a classical problem that is closely related (and in special cases is equivalent) to the famous Uniformization Problem of Klein and Poincaré. Actually, in 1898 Poincaré [11] solved this problem for the simplest case of *parabolic* singularities. Below we formulate his result for the particular case of the standard 2-sphere realized as the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Consider the punctured surface $X = \widehat{\mathbb{C}} \setminus \{z_1, \dots, z_n\}$ with $n \geq 3$ (by applying an appropriate Möbius transformation we can always assume that $z_{n-2} = 0, z_{n-1} = 1, z_n = \infty$). Then the Liouville equation

$$\varphi_{z\bar{z}} = \frac{1}{2} e^{\varphi}$$

(where subscripts stand for the corresponding partial derivatives) has a unique (real-valued) solution φ on X with the following asymptotics:

$$\varphi(z) = \begin{cases} -2\log|z - z_i| - 2\log|\log|z - z_i|| + O(1) & \text{as } z \to z_i,\ i \neq n, \\ -2\log|z| - 2\log\log|z| + O(1) & \text{as } z \to \infty \end{cases}$$

(such a singularity is called parabolic). Geometrically, the Liouville equation means that the conformal metric $ds^2 = e^{\varphi}|dz|^2$ on X has constant negative curvature -1 (that is, hyperbolic), and the above asymptotics of φ guarantee that ds^2 is complete and the area of X is $2\pi(n-2)$.

Received by the editors March 12, 2002.
2000 *Mathematics Subject Classification.* Primary 14H15; Secondary 30F45, 81T40.
Key words and phrases. Fuchsian differential equations, accessory parameters, Liouville action, Weil-Petersson metric.
Research of the first author was partially supported by the NSF grant DMS-9802574.

Poincaré used this result to prove the uniformization theorem, i.e., to show that there exists a complex-analytic covering of the Riemann surface X by the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \operatorname{Im} z > 0\}$. He introduced the quantity

$$T_\varphi = \varphi_{zz} - \tfrac{1}{2}\,\varphi_z^2$$

and showed that when φ satisfies the Liouville equation with parabolic singularities, then T_φ is a meromorphic function on $\widehat{\mathbb{C}}$ of the form

$$T_\varphi(z) = \sum_{i=1}^{n-1} \left(\frac{1}{2(z-z_i)^2} + \frac{c_i}{z-z_i} \right),$$

with the asymptotics

$$T_\varphi(z) = \frac{1}{2z^2} + \frac{c_n}{z^3} + O\left(\frac{1}{z^4}\right) \text{ as } z \to \infty.$$

The coefficients c_i are the famous accessory parameters. They satisfy three obvious linear relations imposed by the asymptotic behaviour of T_φ at ∞. The coefficients $c_1, \dots, c_n$ are uniquely characterized by the fact that the monodromy group of the Fuchsian differential equation

$$\frac{d^2u}{dz^2} + \frac{1}{2}\,T_\varphi(z)u = 0$$

is conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to the group of deck transformations of a covering $\mathbb{H} \to X$.

These ideas of Poincaré were in the spotlight once again about 20 years ago due to Polyakov's path integral formulation of the bosonic string [12] and the conformal field theory of Belavin-Polyakov-Zamolodchikov [2]. Briefly, in the quantum Liouville theory the quantity T_φ plays the role of the $(2,0)$-component of the stress-energy tensor that satisfies conformal Ward identities reflecting conformal symmetry of the theory. At the semi-classical level, as it was first observed by Polyakov, the Ward identity establishes (at the physical level of rigor) a non-trivial relation between the accessory parameters and the critical value of the Liouville action functional (see [13] for details).

In our paper [16], we rigorously proved Polyakov's conjecture using the Ahlfors-Bers theory of quasiconformal mappings and derived simple explicit formulas connecting the Liouville equation with accessory parameters and the Weil-Petersson metric on Teichmüller space. More specifically, let

$$\mathcal{Z}_n = \big\{(z_1, \dots, z_{n-3}) \in \mathbb{C}^{n-3} \,|\, z_i \neq 0, 1 \text{ and } z_i \neq z_k \text{ for } i \neq k\big\}$$

be the configuration space of singular points ($\mathcal{Z}_n$ is isomorphic to the moduli space $\mathcal{M}_{0,n}$ of n-pointed rational curves over $\mathbb{C}$). Then there exists a smooth function $S : \mathcal{Z}_n \to \mathbb{R}$ (critical value of the Liouville action functional; cf. Section 3) such that

$$\text{(I)} \qquad c_i = -\frac{1}{2\pi}\frac{\partial S}{\partial z_i}, \qquad i = 1, \dots, n-3,$$

and

$$\text{(II)} \qquad \frac{\partial c_i}{\partial \bar{z}_k} = \frac{1}{2\pi}\left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_k} \right\rangle_{WP}, \qquad i, k = 1, \dots, n-3,$$

where $\langle\ ,\ \rangle_{WP}$ denotes the Weil-Petersson metric on $\mathcal{Z}_n \cong \mathcal{M}_{0,n}$.[1] An immediate corollary of (I) and (II) is that the critical value S of the Liouville action is a potential for the Weil-Petersson metric.[2]

Although our methods generalize verbatim to hyperbolic 2-spheres with *elliptic* singularities of finite order (in which case there exists a ramified covering $\mathbb{H} \to \widehat{\mathbb{C}}$ branched over singular points $z_1, \dots, z_n$), they no longer work for conical singularities of general type (see Section 2 for precise definitions). However, exact analogs of formulas (I) and (II) hold in this general case as well, provided the orders $\{\alpha_1, \dots, \alpha_n\}$ of singularities $z_1, \dots, z_n$ satisfy some rather mild natural conditions. Physical consideration based on semi-classical limits of conformal Ward identities also suggests the validity of these formulas in a general situation.

The objective of this paper is to give straightforward proofs of (I)-(II) in the case of hyperbolic 2-spheres with conical singularities of general type. Section 2 contains the definitions and background material about the classical Liouville equation, including detailed asymptotics of its solution. In Section 3 we present the action functional for the Liouville equation, introduced in [14], and prove an analogue of formula (I), Theorem 1.[3] In Section 4 we prove an analogue of formula (II) that relates accessory parameters to certain Kähler metrics on the moduli space $\mathcal{M}_{0,n}$ similar to the Weil-Petersson metric — Theorem 2. It is worth noting that the proofs are considerably simpler than those in [16] and do not use Teichmüller theory.

2. Background material

Consider the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with $n \geq 3$ distinct marked points $z_1, \dots, z_n$. As in the Introduction, we normalize the last three points to be 0, 1 and ∞ respectively; so in the sequel we will always assume that $z_{n-2} = 0, z_{n-1} = 1, z_n = \infty$. Let $\alpha = \{\alpha_1, \dots, \alpha_n\}$ be a set of real numbers such that $\alpha_i < 1,\ i = 1, \dots, n$, and

$$\sum_{i=1}^{n} \alpha_i > 2. \tag{1}$$

According to the classical result of Picard [9], [10] (see also [8] and, for a modern proof, [15])[4] there exists a unique conformal metric of constant curvature -1, or the *hyperbolic metric*, on $\widehat{\mathbb{C}}$ with conical singularities of order α_i at $z_i,\ i = 1, \dots, n$. Precisely, it means that such a metric has the form $ds^2 = e^{\varphi}|dz|^2$, where φ is a smooth function on $X = \mathbb{C} \setminus \{z_1, \dots, z_{n-1}\}$ satisfying the Liouville equation

$$\varphi_{z\bar{z}} = \tfrac{1}{2}\, e^{\varphi} \tag{2}$$

and having the following asymptotics near the singular points:

$$\varphi(z) = \begin{cases} -2\alpha_i \log|z - z_i| + O(1) & \text{as } z \to z_i,\ i \neq n, \\ -2(2 - \alpha_n)\log|z| + O(1) & \text{as } z \to \infty. \end{cases} \tag{3}$$

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[1]In [17] we formulated and proved analogs of (I)-(II) for compact Riemann surfaces of arbitrary genus.

[2]These results were used by the second author in the study of the asymptotic behaviour of accessory parameters for degenerating Riemann surfaces [18].

[3]A recent physicists' paper [4] gives a different, computationally more involved proof of Theorem 1.

[4]It is very instructive to compare the approaches of [9], [10], [8] and [15].

The point z_i is then called a *conical singularity* of order α_i, or of angle $\theta_i = 2\pi(1-\alpha_i)$ (we have $\theta_i > 2\pi$ when $\alpha_i < 0$).

Remark 1. If $\alpha_i = 1$, then z_i is a parabolic point, or *cusp* (conical singularity of zero angle), and the asymptotics (3) should be replaced by the one mentioned in the Introduction.

The configuration space $\mathcal{Z}_n$ of singular points is an open subset in $\mathbb{C}^{n-3}$:

$$\mathcal{Z}_n = \left\{(z_1,\dots,z_{n-3}) \in \mathbb{C}^{n-3} \mid z_i \neq 0,1 \text{ and } z_i \neq z_k \text{ for } i \neq k\right\}$$

and is isomorphic to the moduli space $\mathcal{M}_{0,n}$ of n-pointed rational curves over $\mathbb{C}$. For any fixed set of orders α the solution φ to the Liouville equation makes sense as a function of $n-2$ complex variables $z, z_1, \dots, z_{n-3}$, defined on the space

$$\mathcal{Z}_{n+1} = \left\{(z, z_1,\dots,z_{n-3}) \in \mathbb{C}^{n-2} \mid z, z_i \neq 0,1;\ z \neq z_i;\ z_i \neq z_k \text{ for } i \neq k\right\}.$$

The space $\mathcal{Z}_{n+1}$ is fibered over $\mathcal{Z}_n$ by "forgetting" the first coordinate z: the fiber over a point $(z_1,\dots,z_{n-3}) \in \mathcal{Z}_n$ is the surface $\mathbb{C}\setminus\{z_1,\dots,z_{n-3},0,1\}$. It follows from the results of [10], [8], [15] that φ is a real-analytic function on $\mathcal{Z}_{n+1}$.

The $(2,0)$-component of the stress-energy tensor in the Liouville theory is given by the expression

$$T_\varphi = \varphi_{zz} - \tfrac{1}{2}\varphi_z^2. \tag{4}$$

The following result is classical.

Lemma 1. *Let φ be the solution to the Liouville equation with conical singularities* (3). *Then T_φ is a meromorphic function on $\widehat{\mathbb{C}}$ with second-order poles at $z_1,\dots,z_n$. Explicitly,*

$$T_\varphi(z) = \sum_{i=1}^{n-1}\left(\frac{h_i}{2(z-z_i)^2} + \frac{c_i}{z-z_i}\right) \tag{5}$$

and

$$T_\varphi(z) = \frac{h_n}{2z^2} + \frac{c_n}{z^3} + O\left(\frac{1}{z^4}\right) \quad \text{as } z \to \infty, \tag{6}$$

where $h_i = \alpha_i(2-\alpha_i)$, $i = 1,\dots,n$.[5]

Complex numbers c_i are called *accessory parameters*. They are uniquely determined by the singular points $z_1,\dots,z_n$ and the set of orders α. Formula (6) imposes three linear equations on the parameters $c_1,\dots,c_n$:

$$\sum_{i=1}^{n-1} c_i = 0, \qquad \sum_{i=1}^{n-1}(h_i + 2c_i z_i) = h_n, \qquad \sum_{i=1}^{n-1}(h_i z_i + c_i z_i^2) = c_n,$$

so that c_{n-2}, c_{n-1} and c_n are explicit linear combinations of $c_1,\dots,c_{n-3}$ with coefficients depending on z_i and α_i. Real analyticity of φ implies that the accessory parameters are also real-analytic functions on $\mathcal{Z}_n$.

To study the behaviour of φ near the singular points more thoroughly, consider the Fuchsian differential equation

$$\frac{d^2u}{dz^2} + \frac{1}{2}T_\varphi(z)u = 0, \tag{7}$$

[5] The coefficients h_i are conformal weights in quantum Liouville theory [14].

with regular singular points at $z_1, \dots, z_n$. A classical result (see, e.g. [11]), which follows from the fact that $e^{-\varphi/2}$ is a solution to (7), asserts that the monodromy group Γ of the differential equation (7) is, up to a conjugation in $\mathrm{PSL}(2,\mathbb{C})$, a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ (see, e.g., [6], [3], or [7]).[6] Such a group Γ is discrete in $\mathrm{PSL}(2,\mathbb{R})$ if and only if $\alpha_i = 1 - 1/l_i$ for all $i = 1, \dots, n$, where l_i is a positive integer or ∞.

In case of general conical singularities the monodromy group Γ is no longer discrete in $\mathrm{PSL}(2,\mathbb{R})$. It is generated by local monodromies around regular singular points z_i, which, in general, are elliptic elements γ_i of infinite order. If we denote the fixed points of γ_i by $w_i, \bar{w}_i$, then

$$\frac{\gamma_i(z) - w_i}{\gamma_i(z) - \bar{w}_i} = \lambda_i \, \frac{z - w_i}{z - \bar{w}_i}, \quad i = 1, \dots, n,$$

where $\lambda_i = e^{2\pi\sqrt{-1}(1-\alpha_i)}$ is called the multiplier of γ_i.

Remark 2. It is an outstanding problem to find a geometric meaning of the monodromy group Γ in the case of general conical singularities, thus providing another interpretation for the accessory parameters. Perhaps this problem should be considered in the context of A. Connes's [5] non-commutative differential geometry, where such group actions naturally appear.

Let $w = u_1/u_2$ be the ratio of two linearly independent solutions u_1, u_2 of the differential equation (7). It is a multi-valued meromorphic function on $\widehat{\mathbb{C}}$ with ramification points $z_1, \dots, z_n$, and it is single-valued on the universal cover of $X = \widehat{\mathbb{C}} \setminus \{z_1, \dots, z_n\}$. It is a classical result of Schwarz that

$$T_\varphi = \mathcal{S}(w) \tag{8}$$

on X, where $\mathcal{S}(w)$ denotes the Schwarzian derivative of w:

$$\mathcal{S}(w) = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2 .$$

Next, normalize u_1, u_2 in such a way that the monodromy group Γ of (7) is a subgroup of $\mathrm{PSU}(1,1)$. The multi-valued function w admits the following expansion in the neighborhood of each singular point z_i:

$$\sigma_i(w(z)) = \zeta_i^{1-\alpha_i} \sum_{k=0}^{\infty} a_i^{(k)} \zeta_i^k \quad \text{as } \zeta_i \to 0,\ i = 1, \dots, n. \tag{9}$$

Here ζ_i is a local uniformizer: $\zeta_i = z - z_i$ for $i = 1, \dots, n-1$, and $\zeta_n = 1/z$, and $\sigma_i \in \mathrm{PSU}(1,1)$ diagonalizes local monodromy γ_i around z_i, $i = 1, \dots, n$. Moreover, the coefficients $a_i^{(k)}$ are (locally) real-analytic on $\mathcal{Z}_n$, as follows from the analytic dependence on parameters of solutions to ordinary differential equations.

Lemma 2. *The solution φ to the Liouville equation* (2) *with conical singularities* (3) *is given by the formula*

$$e^\varphi = \frac{4|w'|^2}{(1 - |w|^2)^2}\,,$$

where $w = u_1/u_2$, and u_1, u_2 are two linearly independent solutions of the Fuchsian differential equation (7) *with monodromy in* $\mathrm{PSU}(1,1)$.

[6] Among many available references, [6] is classical, [3] gives a detailed exposition of Fuchsian differential equations, and [7] is a modern introduction to the subject.

Proof. Since the monodromy is in $\mathrm{PSU}(1,1)$, the function $\log\left(\frac{4|w'|^2}{(1-|w|^2)^2}\right)$ is real and single-valued on X. Moreover, it is easy to check that this function satisfies the Liouville equation, and by (9) it has the same asymptotics (3) as φ. Therefore, it must be equal to φ. $\square$

Remark 3. When $\alpha_i = 1$, $i = 1, \dots, n$, it is more convenient to normalize solutions u_1, u_2 so that $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ (see [16]).

From the equality (8) and expansions (9) we readily get the following formula for the accessory parameters (cf. Lemma 1 in [16]).

Lemma 3.

$$c_i = \frac{h_i}{1-\alpha_i} \cdot \frac{a_i^{(1)}}{a_i^{(0)}}, \quad i = 1, \dots, n,$$

where $h_i = \alpha_i(2 - \alpha_i)$.

Finally, we summarize all the necessary facts about the asymptotic behaviour of φ and its derivatives in the next statement (cf. Lemma 2 in [16]).

Lemma 4. *The solution φ to the Liouville equation* (2) *with conical singularities* (3) *has the following asymptotic expansions near the singular points $z = z_i$, uniform in a neighborhood of $(z_1, \dots, z_{n-3})$ in $\mathcal{Z}_n$:*

(i)

$$\varphi_z(z) = \begin{cases} -\dfrac{\alpha_i}{\zeta_i} + \dfrac{c_i}{\alpha_i} + \dfrac{f_i(|\zeta_i|)}{\zeta_i} + o(1) & \text{as } z \to z_i,\ i \neq n, \\ -(2-\alpha_n)\zeta_n - \dfrac{c_n}{\alpha_n} \cdot \zeta_n^2 + f_n(|\zeta_n|)\zeta_n + o\left(|\zeta_n|^2\right) & \text{as } z \to \infty, \end{cases}$$

where $\zeta_i = z - z_i$ $(i \neq n)$ and $\zeta_n = 1/z$ are local coordinates near the singular points, and

$$f_i(t) = O\left(t^{2(1-\alpha_i)}\right) \quad \text{as } t \to 0,\ i = 1, \dots, n.$$

(ii) *For $i = 1, \dots, n-3$*

$$\varphi_{zz}(z) = \frac{\alpha_i + g_i^{(0)}(\zeta_i) + \zeta_i g_i^{(1)}(\zeta_i)}{\zeta_i^2} + O(1),$$

where

$$g_i^{(0)}(t),\ g_i^{(1)}(t) = O\left(t^{2(1-\alpha_i)}\right) \quad \text{as } t \to 0.$$

(iii) *For $i = 1, \dots, n-3$ and $k = 1, \dots, n$, there exist constants d_{ik} such that*

$$\varphi_{z_i}(z) = \begin{cases} -\delta_{ik}\varphi_z(z) + d_{ik} + o(1) & \text{as } z \to z_k,\ k \neq n, \\ \ \ d_{in} + o(1) & \text{as } z \to \infty. \end{cases}$$

(iv) *If $\alpha_k > 0$ for each $k = 1, \dots, n$, then for $i = 1, \dots, n-3$,*

$$-2e^{-\varphi}\varphi_{z_i\bar{z}} = \begin{cases} \delta_{ik} + O\left(|z - z_k|^{\,\min\{1,\,2\alpha_k\}}\right) & \text{as } z \to z_k,\ k \neq n, \\ O\left(|z|^{\,\max\{1,\,2(1-\alpha_n)\}}\right) & \text{as } z \to \infty. \end{cases}$$

Proof. Parts (i)-(iii) follow from (9) and Lemmas 2 and 3; part (iv) follows from (i), (iii), the Liouville equation (2) and asymptotics (3). Uniform estimates for the remainder terms follow from the real analyticity of the coefficients $a_i^{(k)}$ as functions of $z_1, \dots, z_{n-3}$. One can also prove (i)-(iv) directly from the Liouville equation and asymptotics (3) by observing that the solution φ admits the following expansion in a neighborhood of each z_i:

$$\varphi(z) = -2\alpha_i \log|z - z_i| + \xi^{(0)}(z) + \sum_{k=1}^{\infty} |z - z_i|^{2k(1-\alpha_i)} \xi^{(k)}(z),\ i = 1, \dots, n-1,$$

and a similar expansion at ∞, where $\xi^{(k)}(z)$ are real-analytic as functions on the fibered space $\mathcal{Z}_{n+1}$ (real-analytic dependence on $z_1, \dots, z_{n-3}$ follows from the analysis in [9], [10], [8], [15]). □

3. Liouville action and accessory parameters

For a given set of orders $\alpha = \{\alpha_1, \dots, \alpha_n\}$ the action functional for the Liouville equation (2) is defined in [14] by the formula

$$S_\alpha[\psi] = \lim_{\varepsilon \to 0} S_\alpha^\varepsilon[\psi], \tag{10}$$

where

$$\begin{aligned} S_\alpha^\varepsilon[\psi] = &\iint\limits_{X^\varepsilon} (|\psi_z|^2 + e^\psi) \left| \frac{dz \wedge d\bar{z}}{2} \right| \\ &+ \frac{\sqrt{-1}}{2} \sum_{i=1}^{n-1} \alpha_i \oint\limits_{C_i^\varepsilon} \psi \left(\frac{d\bar{z}}{\bar{z} - \bar{z}_i} - \frac{dz}{z - z_i} \right) \\ &+ \frac{\sqrt{-1}}{2} (2 - \alpha_n) \oint\limits_{C_n^\epsilon} \psi \left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z} \right) \\ &- 2\pi \sum_{i=1}^{n-1} \alpha_i^2 \log \varepsilon - 2\pi (2 - \alpha_n)^2 \log \varepsilon. \end{aligned} \tag{11}$$

Here $X^\varepsilon = \mathbb{C} \setminus \left(\bigcup_{i=1}^{n-1} \{|z - z_i| < \varepsilon\} \cup \{|z| > 1/\varepsilon\} \right)$, and the circles $C_i^\varepsilon = \{|z - z_i| = \varepsilon\}$, $i = 1, \dots, n-1$, and $C_n^\varepsilon = \{|z| = 1/\varepsilon\}$ are oriented as the boundary components of X^ε.

The functional S_α is well-defined on the space $\mathcal{CM}_\alpha$ of all conformal metrics $e^\psi |dz|^2$ on $\widehat{\mathbb{C}}$ with conical singularities at $z_1, \dots, z_n$ of orders $\alpha_1, \dots, \alpha_n$, satisfying

$$\psi_z(z) = \begin{cases} -\dfrac{\alpha_i}{z - z_i} \left(1 + O\left(|z - z_i|^{\min\{1,\, 2(1-\alpha_i)\}}\right)\right) & \text{as } z \to z_i,\ i \neq n, \\ -(2 - \alpha_n)\, \dfrac{1}{z} \left(1 + O\left(|z|^{-\min\{1,\, 2(1-\alpha_n)\}}\right)\right) & \text{as } z \to \infty. \end{cases} \tag{12}$$

Remark 4. The Liouville equation is the Euler-Lagrange equation for the functional S_α. Indeed, the contour integrals in (11) ensure that for any $e^\psi |dz|^2 \in \mathcal{CM}_\alpha$ and $u \in C^\infty(\widehat{\mathbb{C}}, \mathbb{R})$,

$$\lim_{t \to 0} \frac{S[\psi + tu] - S[\psi]}{t} = \iint\limits_{\mathbb{C}} (-2\psi_{z\bar{z}} + e^\psi)\, u\, \frac{|dz \wedge d\bar{z}|}{2},$$

where the integral on the right-hand side is convergent. Thus the functional S_α has a non-degenerate critical point given by the hyperbolic metric.

The Liouville action evaluated on the solution φ to the Liouville equation is a real-valued function $S_\alpha[\varphi] = S_\alpha(z_1, \dots, z_{n-3})$ on the configuration space $\mathcal{Z}_n$ depending on $\alpha_1, \dots, \alpha_n$ as parameters.

Theorem 1. *For any fixed set of orders $\alpha = \{\alpha_1, \dots, \alpha_n\}$ such that $\alpha_i < 1$ and $\sum_{i=1}^n \alpha_i > 2$, the function $S_\alpha : \mathcal{Z}_n \longrightarrow \mathbb{R}$ is differentiable and*

$$c_i = -\frac{1}{2\pi}\frac{\partial S_\alpha}{\partial z_i}, \qquad i = 1, \dots, n-3, \tag{13}$$

where c_i are the accessory parameters defined by (5).

Proof. First we show that

$$\lim_{\varepsilon \to 0} \frac{\partial S_\alpha^\varepsilon}{\partial z_i} = -2\pi c_i \tag{14}$$

pointwise on the configuration space $\mathcal{Z}_n$. We have

$$\begin{aligned} \frac{\partial S_\alpha^\varepsilon}{\partial z_i} = \frac{\sqrt{-1}}{2} & \left(\iint\limits_{X^\varepsilon} \frac{\partial}{\partial z_i}(|\,\varphi_z|^2 + e^\varphi)\, dz \wedge d\bar{z} + \oint\limits_{C_i^\varepsilon} (|\,\varphi_z|^2 + e^\varphi)\, d\bar{z} \right) \\ & + \frac{\sqrt{-1}}{2} \sum_{k=1}^{n-1} \alpha_k \oint\limits_{C_k^\varepsilon} (\varphi_{z_i} + \delta_{ik}\varphi_z) \left(\frac{d\bar{z}}{\bar{z} - \bar{z}_k} - \frac{dz}{z - z_k} \right) \\ & + \frac{\sqrt{-1}}{2} (2 - \alpha_n) \oint\limits_{C_n^\varepsilon} \varphi_{z_i} \left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z} \right). \end{aligned} \tag{15}$$

Using part (i) of Lemma 4, we see that

$$\frac{\sqrt{-1}}{2} \oint\limits_{C_i^\varepsilon} |\,\varphi_z|^2\, d\bar{z} \longrightarrow \pi c_i \qquad \text{as } \varepsilon \to 0.$$

From the Liouville equation we get

$$\oint\limits_{C_i^\varepsilon} e^\varphi d\bar{z} = -\frac{1}{2} \oint\limits_{C_i^\varepsilon} \varphi_{zz} dz,$$

which tends to 0 as $\varepsilon \to 0$ because of part (ii) of Lemma 4. It follows from part (iii) of Lemma 4 that the contour integrals in the second and third lines of (15) tend to

$$-2\pi c_i - 2\pi \sum_{k=1}^{n-1} \alpha_k d_{ik} - 2\pi(\alpha_n - 2) d_{in}$$

as $\varepsilon \to 0$. An obvious identity,

$$\frac{\partial}{\partial z_i} |\varphi_z|^2 dz \wedge d\bar{z} = d\,(\varphi_{z_i}\varphi_{\bar{z}}\, d\bar{z} - \varphi_{z_i}\varphi_z\, dz) - 2\varphi_{z_i}\varphi_{z\bar{z}}\, dz \wedge d\bar{z},$$

combined with the Liouville equation yields the following simple formula:

$$\frac{\partial}{\partial z_i}(|\varphi_z|^2 + e^\varphi)\, dz \wedge d\bar{z} = d\,(\varphi_{z_i}\varphi_{\bar{z}}\, d\bar{z} - \varphi_{z_i}\varphi_z\, dz). \tag{16}$$

This reduces the area integral in (15) to a sum of contour integrals. These contour integrals are again easy to evaluate using parts (i) and (iii) of Lemma 4, and all together they tend to

$$-\pi c_i + 2\pi \sum_{k=1}^{n-1} \alpha_k d_{ik} + 2\pi(\alpha_n - 2)d_{in}$$

as $\varepsilon \to 0$. Adding all the terms on the right-hand side of (15), we get $-2\pi c_i$ in the limit as $\varepsilon \to 0$. Finally, we observe that the convergence of (14) is uniform on compact subsets of $\mathcal{Z}_n$ because so are the estimates in Lemma 4. □

Remark 5. The same method works for $\alpha_i = 1$, $i = 1, \ldots, n$. In this case, formula (11) for the functional $S^\varepsilon[\varphi]$ contains an additional regularizing term $4\pi(n-2)\log|\log\varepsilon|$. By part 2) of Lemma 2 in [16], no contour integrals contribute to the classical action $S[\varphi]$. This gives a much simpler proof of Theorem 1 in [16] along the lines of this paper without using either the uniformization theorem or the quasiconformal mappings.

4. Accessory parameters and Kähler metrics on $\mathcal{M}_{0,n}$

Throughout this section we assume, in addition, that the orders $\alpha_1, \ldots, \alpha_n$ are all positive,[7] i.e., $\alpha_i \in (0,1)$ for each $i = 1, \ldots, n$, and $\sum_{i=1}^n \alpha_i > 2$. To every such set of orders $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ we associate a Hermitian metric on the configuration space $\mathcal{Z}_n \cong \mathcal{M}_{0,n}$ as follows.

Consider the kernel

$$R(\zeta, z) = -\frac{1}{\pi}\left(\frac{1}{\zeta - z} + \frac{z-1}{\zeta} - \frac{z}{\zeta - 1}\right), \qquad (\zeta, z) \in \mathbb{C} \times \mathbb{C}, \tag{17}$$

and put

$$Q_i(z) = R(z, z_i), \quad i = 1, \ldots, n-3. \tag{18}$$

Clearly, the functions Q_i are linearly independent. It follows from the positivity of orders α_i and (3) that the functions Q_i are square-integrable on $\widehat{\mathbb{C}}$ with respect to the measure $e^{-\varphi}\frac{|dz\wedge d\bar{z}|}{2}$. We define the scalar products of the basis of 1-forms on $\mathcal{Z}_n$ over the point $(z_1, \ldots, z_{n-3}) \in \mathcal{Z}_n$ by the formula

$$(dz_i, dz_k)_\alpha = \iint\limits_{\mathbb{C}} Q_i \overline{Q}_k e^{-\varphi}\, \frac{|dz \wedge d\bar{z}|}{2}, \quad i, k = 1, \ldots, n-3. \tag{19}$$

The scalar products $\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_k}\rangle_\alpha$ are given by the elements of the inverse matrix to $\{(dz_i, dz_k)_\alpha\}_{i,\,k=1}^{n-3}$. Since the matrix $\{\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_k}\rangle_\alpha\}_{i,\,k=1}^{n-3}$ is non-degenerate and depends real analytically on z_i, it gives rise to a Hermitian metric on $\mathcal{Z}_n$ which we denote by $\langle\,\cdot\,,\cdot\rangle_\alpha$. This metric is analogous to the celebrated Weil-Petersson metric on the moduli space $\mathcal{M}_{0,n}$.[8]

Remark 6. In Teichmüller theory, when all $\alpha_i = 1$, the holomorphic cotangent space to $\mathcal{Z}_n$ at the point $(z_1, \ldots, z_{n-3})$ is identified by means of quasiconformal mappings with the space of rational functions on $\widehat{\mathbb{C}}$ with only simple poles at $z_1, \ldots, z_{n-3}, 0, 1, \infty$, and dz_i then corresponds to Q_i (see, e.g., [16] and references therein). Here we use the same identification directly.

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[7] This is equivalent to the condition that all conformal weights h_i are positive.

[8] We get the Weil-Petersson metric if all the orders α_i are equal to 1.

The kernel R, roughly speaking, inverts the operator $\partial/\partial\bar{z}$ on $\mathbb{C}$. The precise statement (see, e.g., [1] for details) is essentially a version of the Pompeiu formula.

Lemma 5. *Let g be a locally integrable function on $\mathbb{C}$ such that $g(z) = o(|z|)$ as $z \to \infty$. Then the equation*

$$f_{\bar{z}} = g$$

has a unique solution on $\mathbb{C}$ satisfying $f(0) = f(1) = 0$ and $f(z) = o(|z|^2)$ as $z \to \infty$. This solution is explicitly given by the formula

$$f(z) = \iint\limits_{\mathbb{C}} g(\zeta) R(\zeta, z) \, \frac{|d\zeta \wedge d\bar{\zeta}|}{2}. \tag{20}$$

Let us formulate the main result of this section.

Theorem 2. *For any set of orders $\alpha = \{\alpha_1, \dots, \alpha_n\}$ such that $\alpha_i \in (0,1)$ for each $i = 1, \dots, n$ and $\sum_{i=1}^{n} \alpha_i > 2$, we have*

$$\frac{\partial c_i}{\partial \bar{z}_k} = \frac{1}{2\pi} \left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_k} \right\rangle_{\alpha}, \qquad i, k = 1, \dots, n-3. \tag{21}$$

Proof. As we mentioned in Section 2, the accessory parameters $c_1, \dots, c_{n-3}$ are real-analytic functions on $\mathcal{Z}_n$. Now consider the functions

$$F^i = -2e^{-\varphi} \varphi_{z_i \bar{z}}, \qquad i = 1, \dots, n-3.$$

According to part (iv) of Lemma 4 we have

$$\begin{aligned} F^i(z_k) &= \delta_{ik}, \qquad k = 1, \dots, n-1, \\ F^i(z) &= O(|z|^{\max\{1,\, 2(1-\alpha_n)\}}), \qquad z \to \infty. \end{aligned} \tag{22}$$

Moreover, as follows from (4) and (5),

$$\begin{aligned} F^i_{\bar{z}} &= 2e^{-\varphi} \varphi_{\bar{z}} \, \varphi_{z_i \bar{z}} - 2e^{-\varphi} \varphi_{z_i \bar{z}\bar{z}} = -2e^{-\varphi} \frac{\partial}{\partial z_i} \left(\varphi_{\bar{z}\bar{z}} - \frac{1}{2} \varphi_{\bar{z}}^2 \right) \\ &= -2e^{-\varphi} \sum_{k=1}^{n-1} \frac{1}{\bar{z} - \bar{z}_k} \cdot \frac{\partial \bar{c}_k}{\partial z_i} = 2\pi e^{-\varphi} \sum_{k=1}^{n-3} \frac{\partial \bar{c}_k}{\partial z_i} \, \overline{Q}_k. \end{aligned}$$

Lemma 5 applied to $g = F^i_{\bar{z}}$ yields

$$F^i(z) = \iint_{\mathbb{C}} F^i_{\bar{\zeta}}(\zeta) R(\zeta, z) \, \frac{|d\zeta \wedge d\bar{\zeta}|}{2}, \qquad i = 1, \dots, n-3.$$

Putting $z = z_j$ and using (22) we get that

$$\delta_{ij} = 2\pi \sum_{k=1}^{n-3} \frac{\partial \bar{c}_k}{\partial z_i} \, (dz_j, dz_k)_{\alpha}, \qquad i, j = 1, \dots, n-3,$$

which proves the theorem. $\square$

Remark 7. The same arguments prove Theorem 2 in [16], making the uniformization theorem and quasiconformal mappings redundant also in the case when all $\alpha_i = 1$.

Corollary 1. *For any set α as in Theorem 2,*

$$\left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_k} \right\rangle_{\alpha} = -\frac{\partial^2 S_\alpha}{\partial z_i \, \partial \bar{z}_k}, \qquad i, k = 1, \dots, n-3.$$

That is, the metric $\langle \cdot, \cdot \rangle_\alpha$ is Kähler and the function $-S_\alpha$ is its real-analytic Kähler potential on $\mathcal{Z}_n$.

Proof. Immediately follows from Theorems 1 and 2. □

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